บทความวิจัย

การประมาณค่าพารามิเตอร์ด้วยวิธีแอนเดอร์สัน-ดาร์ลิง สำหรับการแจกแจงแบบมาร์แชลล์โอคินความยาวเอนเอียงเลขชี้กำลัง และการประยุกต์ใช้กับข้อมูลทางด้านวิศวกรรมและสิ่งแวดล้อม

ดาริกา แย้มรับบุญ และ อารียา สุดสุข*
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

* ผู้นิพนธ์ประสานงาน โทรศัพท์ 0 5596 3212 อีเมล: areeyas@nu.ac.th DOI: 10.14416/j.kmutnb.2022.09.016
รับเมื่อ 7 มิถุนายน 2564 แก้ไขเมื่อ 29 กันยายน 2564 ตอบรับเมื่อ 5 ตุลาคม 2564 เผยแพร่ออนไลน์ 27 กันยายน 2565

บทคัดย่อ

การศึกษาครั้งนี้มีวัตถุประสงค์เพื่อประยุกต์ใช้ตัวประมาณแอนเดอร์สัน-ดาร์ลิง (Minimum Anderson-Darling Estimator; MADE) ในการประมาณค่าพารามิเตอร์ที่ไม่ทราบค่าของการแจงแจงแบบมาร์แชล์โอคินความยาวเอนเอียงเลขชี้กำลัง (Marshall-Olkin Length-Biased Exponential (MOLBE) Distribution) ซึ่งการแจกแจงนี้เป็นการการแจกแจงที่ขยายจากการแจงแจงแบบความยาวเอนเอียงเลขชี้กำลัง (Length-Biased Exponential Distribution) โดยใช้หลักการของมาร์แชลโอคิน (Marshall-Olkin Scheme) อีกทั้งการแจกแจง MOLBE ยังเป็นการแจกแจงที่มีความยืดหยุ่นเป็นอย่างมากในการนำไปใช้อธิบายข้อมูลจริงที่มีลักษณะเบ้ขวา นอกจากนี้ประสิทธิผลของตัวประมาณ MADE ที่ใช้ในการประมาณค่าพารามิเตอร์ของการแจกแจง MOLBE ถูกนำเสนอผ่านข้อมูลจริงทางด้านวิศวกรรมและสิ่งแวดล้อมที่ไม่ถูกเซ็นเซอร์จำนวน 3 ชุดข้อมูล ซึ่งชุดข้อมูลที่นำมาใช้ประกอบไปด้วยข้อมูลที่มีหลากหลายรูปแบบ เช่น เบ้ซ้าย เบ้ขวา และสมมาตร

คำสำคัญ: การประมาณค่าพารามิเตอร์ด้วยวิธีแอนเดอร์สัน-ดาร์ลิง การแจกแจงแบบมาร์แชลล์โอคินความยาวเอนเอียงเลขชี้กำลัง วิศวกรรม สิ่งแวดล้อม

การอ้างอิงบทความ: ดาริกา แย้มรับบุญ และ อารียา สุดสุข, “การประมาณค่าพารามิเตอร์ด้วยวิธีแอนเดอร์สัน-ดาร์ลิง สำหรับการแจกแจงแบบมาร์แชลล์โอคินความยาวเอนเอียงเลขชี้กำลัง และการประยุกต์ใช้กับข้อมูลทางด้านวิศวกรรมและสิ่งแวดล้อม,” *วารสารวิชาการพระจอมเกล้าพระนครเหนือ*, ปีที่ 33, ฉบับที่ 3, หน้า 1–13, เลขที่บทความ 233-095128, ก.ค.–ก.ย. 2566.

Research Article

Anderson-Darling Parameter Estimation of the Marshall-Olkin Length-Biased Exponential Distribution with Applications in Engineering and Environment

Darika Yamrubboon and Areeya Sudsuk*
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, Thailand

* Corresponding Author, Tel. 0 5596 3212, E–mail: areeyas@nu.ac.th DOI: 10.14416/j.kmutnb.2022.09.016
Received 7 June 2021; Revised 29 September 2021; Accepted 5 October 2021; Published online: 27 September 2022

Abstract

This paper aims at applying the minimum Anderson–Darling estimator (MADE) to the Marshall-Olkin Length-Biased Exponential (MOLBE) distribution for estimating its unknown parameters. The MOLBE distribution, which extends the Length-Biased Exponential distribution based on the Marshall-Olkin scheme, seems more flexible in modeling real data with right skewed shape. The efficacy of the MADE for the MOLBE parameter estimation is shown through three uncensored real datasets in engineering and environment with various shapes such as slightly left skewed, right skewed and symmetrical.

Keywords: Minimum Anderson-Darling Estimator, The Marshall-Olkin Length-Biased Exponential Distribution, Engineering, Environment

Please cite this article as: D. Yamrubboon and A. Sudsuk, "Anderson-Darling parameter estimation of the Marshall-Olkin length-biased exponential distribution with applications in engineering and environment," *The Journal of KMUTNB*, vol. 33, no. 3, pp. 1–13, ID. 233-095128, Jul.–Sep. 2023.

1. Introduction

Lifetime data modeling is an important topic in applied sciences including engineering, environment, biomedical science, finance, and insurance. In the field of environment and engineering, for example, a model of lifetime data allows us to predict the occurrence of natural events such as the next ocean waves caused by earth movement. There are many choices of probability distributions that allow us to characterize the lifetime data. The exponential distribution is a commonly used baseline distribution. However, the existing models are not suitable or less representative of actual data in many situations; according to [1]. As a result, developing distributions that can better describe certain phenomena while also being more flexible than the baseline distribution are critical [2].

The moment exponential distribution also known as the length biased exponential (LBE) distribution is an extension of the exponential distribution [3]. Following the concept introduced by Fisher in 1934, the LBE distribution was constructed by assigning weight to the exponential distribution [4]. It was proved that the LBE distribution is more flexible than the exponential distribution as discussed in [5]. In 2021, the parameter estimation of LBE distribution has been studied under simple – step - stress cumulative exposure model. The maximum likelihood (ML) method is used to estimate a parameter in the model; according to [6].

Recently, a new extension of the LBE distribution called the Marshall-Olkin Length-Biased Exponential (MOLBE) distribution was proposed by Haq *et al.* [5]. The MOLBE distribution is a new life time distribution under the Marshall-Olkin (MO) family framework by adding a shape parameter [7]. The mathematical properties of this distribution were derived, including survival and hazard rate functions, ordinary and incomplete moments, generating function, entropies, mean residual life and mean inactivity time. The estimation of the MOLBE parameters based on ML method was also considered and applied in tensile strength of carbon fibers dataset as demonstrated in [5].

The maximum likelihood estimator (MLE) is one of the most widely used baseline methods for estimating the parameters of a model. In recent years, the maximum goodness-of-fit estimators (also called minimum distance estimators) have gained a great interest to practitioners and applied statisticians. Especially in the minimum Anderson–Darling estimator (MADE), it provides a better solution on the case of many location scale distributions. The application of the MADE method to the approximated cumulative distribution function was claimed to be more computationally efficient than the application of the ML method to the approximated probability density function [8]. Furthermore, the MADE was discussed by many authors such as Dey *et al.* [9], ZeinEldin *et al.* [10], and Al-Mofleh *et al.* [11].

Three real datasets in environment and engineering are modelled using MOLBE distribution. In this paper, the MADE is employed to estimate the parameters of the distributions. The contents of this paper are as follows: section 2 provides a brief introduction of the characteristic of MOLBE distribution and the parameter estimation according to the minimum distance estimator. Three real datasets modelled as MOLBE distribution are presented in section 3. Afterward, some of conclusions are drawn in section 4.

2. Materials and Methods

In this section, the MOLBE distribution and its mathematical properties are introduced, as well as its parameters estimation based on the classical method.

2.1 The MOLBE distribution

Haq *et al.* [5] introduced MOLBE distribution as an alternative to LBE distribution to analyze a real life dataset. It is a new extension of the LBE using the MO distribution family. The MOLBE distribution has two parameters, namely scale parameter β and shape parameter γ

Definition. Let X be a random variable that follows the MOBLE distribution with parameters $\beta > 0$ and $\gamma > 0$, written as $X \sim \mathrm{MOLBE}(\beta, \gamma)$ The probability density function (pdf) of X is given by

$$f(x) = \frac{\gamma \dfrac{x}{\beta^2} \exp(-x/\beta)}{\left(1-(1-\gamma)\left(1+\dfrac{x}{\beta}\right)\exp(-x/\beta)\right)^2}, \quad (1)$$

and the cumulative density function (cdf) of X is

$$F(x) = \frac{1-\left(1+\dfrac{x}{\beta}\right)\exp(-x/\beta)}{1-(1-\gamma)\left(1+\dfrac{x}{\beta}\right)\exp(-x/\beta)}, \quad (2)$$

where $x > 0$ for $\beta > 0$, and $\gamma > 0$.

Corollary. If $\gamma = 1$ then the MOLBE distribution reduces to the LBE distribution with pdf shown in Equation (3).

$$f(x) = \frac{x}{\beta^2} \exp(-x/\beta), \quad (3)$$

where $x > 0$ for $\beta > 0$.

In Figure 1, the pdf and cdf plots for the MOLBE with various values of the parameters β and γ are illustrated.

- γ is a shape parameter, since for a fixed value of β, the shape of density function changes as changes, (see Figure 1). The pdf can be unimodal with right-skew. Indeed as γ decreases, the pdf becomes more extremely right skewed in Figure 1 (a) and (e).

- The pdf of MOLBE distribution tends to normal distribution as the value of γ increases, while the value of β is kept constant, as shown in Figure 1 (c).

- From Figure 1 (b), (d), and (f), the cdf of MOLBE distribution satisfies the following properties: $\lim_{x\to\infty} F(x) = 1$ and $F(x)$ is non-decreasing.

2.2 The mathematical properties of the MOLBE distribution

In this part, we present some basic mathematical properties of the MOLBE distribution such as survival and hazard functions, the expansion of the probability density function, moments, and the moment generating function. These properties were derived in Haq *et al.* [5].

2.2.1 Survival and hazard functions

The survival function, $S(x)$, which is known as the reliability function, is probability of surviving beyond a specified time. The hazard function or the hazard rate, $h(x)$, is used to monitor the lifetime of a unit across the support of its life-time distribution. The $S(x)$ and the $h(x)$ of MOLBE distribution are in Equation (4) and Equation (5), respectively.

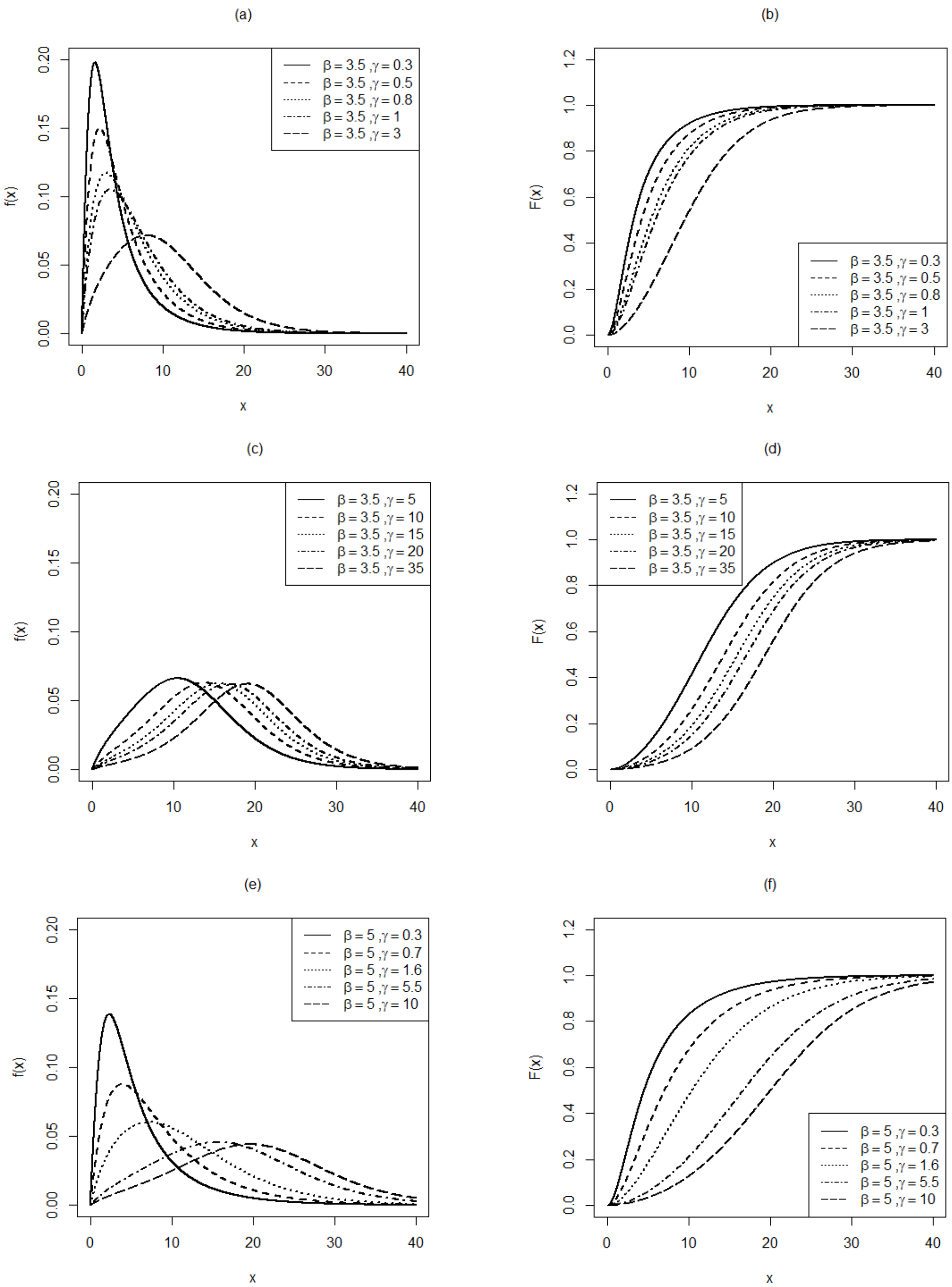

Figure 1 Some plots of pdf and cdf for the MOLBE distribution with some parameter values.

$$S(x)=\frac{\gamma\left(1+\frac{x}{\beta}\right)\exp\left(-x/\beta\right)}{1-(1-\gamma)\left(1+\frac{x}{\beta}\right)\exp\left(-x/\beta\right)} \quad (4)$$

and

$$h(x)=\frac{\frac{x}{\beta^2}}{\left(1+\frac{x}{\beta}\right)\left[1-(1-\gamma)\left(1+\frac{x}{\beta}\right)\exp\left(-x/\beta\right)\right]}. \quad (5)$$

Haq *et al.* [5] conducted a study of the MOLBE hazard function that can be unimodal and increasing.

2.2.2 The expansion of the probability density function

Using both the generalized binomial expansion and power series, the linear representation for the pdf is given by Equation (1) can be expressed in Equation (6).

$$f(x)=\frac{\gamma}{\beta}\sum_{kj=0}^{\infty}\left(k+1\right)\left(1-\gamma\right)^{k}\binom{k}{j}\left(\frac{x}{\beta}\right)^{j+1} \times\exp\left[-\left(k+1\right)x/\beta\right]. \quad (6)$$

2.2.3 Moments

The moments of the MOLBE distribution reveal its many important properties. The rth moment of $X\sim \mathrm{MOLBE}(\beta,\gamma)$ is given as Equation (7).

$$\mu_r'=\gamma\sum_{kj=0}^{\infty}\frac{\left(1-\gamma\right)^{k}\beta^{r}}{\left(k+1\right)^{j+r+1}}\binom{k}{j}\Gamma\left(j+r+2\right). \quad (7)$$

2.2.4 Moments generating function

The moment generating function (mgf) of the MOLBE distribution [5] is given by Equation (8).

$$M_x\left(t\right)=\gamma\sum_{kj=0}^{\infty}\frac{(k+1)\left(1-\gamma\right)^{k}\beta^{r}}{\left(k+1-\beta t\right)^{j+2}}\binom{k}{j}\Gamma\left(j+2\right). \quad (8)$$

The rth moment about the origin can be obtained by differentiating the mgf r times with respect to t and setting $t=0$. The expectation value, variance, skewness and kurtosis can be obtained easily using this method. More details of the properties MOLBE distribution were provided in Haq *et al.* [5].

2.3 Parameters estimation

In previous work, the ML method is used to estimate the unknown parameters of the distribution [5]. In this study we use the minimum distance estimators, a frequentist approach, to estimate the unknown parameters of the distribution. In this approach the unknown parameters of the cdf are obtained by minimizing any empirical distribution.

The Anderson-Darling distance function (ADF) which is applied to goodness-of-fit tests for different distribution types is expressed as Equation (9).

$$A(\boldsymbol{\theta})=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left(F\left(X_{i:n}\mid\boldsymbol{\theta}\right)\right)+\log\left(1-F\left(X_{n+1-i:n}\mid\boldsymbol{\theta}\right)\right)\right\}, \quad (9)$$

where $\boldsymbol{\theta}$ is the vector of parameters in the cdf. The point estimation of MADE were defined in [8], [12], resulting in $A(\hat{\boldsymbol{\theta}})=\min_{\boldsymbol{\theta}}A(\boldsymbol{\theta})$.

The estimator of the parameters β and γ for the MOLBE distribution is obtained from the MADE method. Let $X_1, X_2,\ldots, X_n$, be an ordered independent and identically distributed random sample of size n from the MOLBE population with cdf defined in Equation (2) and the vector of the unknown parameters is denoted as $\boldsymbol{\Theta}=(\beta,\gamma)^T$. Thus, the ADF of MOLBE distribution is given by

$$A(\mathbf{\Theta}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1) \times \left\{ \log\left(\frac{1-\left(1+\frac{x_{(i)}}{\beta}\right)\exp\left(-x_{(i)}/\beta\right)}{1-(1-\gamma)\left(1+\frac{x_{(i)}}{\beta}\right)\exp\left(-x_{(i)}/\beta\right)} \right) + \log\left(1 - \frac{1-\left(1+\frac{x_{(n+1-i)}}{\beta}\right)\exp\left(-x_{(n+1-i)}/\beta\right)}{1-(1-\gamma)\left(1+\frac{x_{(n+1-i)}}{\beta}\right)\exp\left(-x_{(n+1-i)}/\beta\right)} \right) \right\}. \quad (10)$$

Consequently, the Anderson-Darling estimates of $\mathbf{\Theta}$ denoted by $\hat{\mathbf{\Theta}} = (\hat{\gamma}, \hat{\beta})^T$ are obtained by minimizing $A(\mathbf{\Theta})$ in Equation (10) with respect to β and γ. These estimators are calculated by solving the nonlinear equations as follows:

$$\frac{\partial A(\mathbf{\Theta})}{\partial \beta} = \sum_{i=1}^{n}(2i-1)\left\{ \frac{\delta_1\left(X_{(i)}\,|\,\mathbf{\Theta}\right)}{F\left(X_{(i)}\,|\,\mathbf{\Theta}\right)} - \frac{\delta_1\left(X_{(n+1-i)}\,|\,\mathbf{\Theta}\right)}{1-F\left(X_{(n+1-i)}\,|\,\mathbf{\Theta}\right)} \right\} = 0, \quad (11)$$

$$\frac{\partial A(\mathbf{\Theta})}{\partial \gamma} = \sum_{i=1}^{n}(2i-1)\left\{ \frac{\delta_2\left(X_{(i)}\,|\,\mathbf{\Theta}\right)}{F\left(X_{(i)}\,|\,\mathbf{\Theta}\right)} - \frac{\delta_2\left(X_{(n+1-i)}\,|\,\mathbf{\Theta}\right)}{1-F\left(X_{(n+1-i)}\,|\,\mathbf{\Theta}\right)} \right\} = 0, \quad (12)$$

where

$$\delta_1\left(X_{(\cdot)}\,|\,\mathbf{\Theta}\right) = \frac{\frac{\gamma}{\beta^2}x_{(\cdot)}\exp\left(-x_{(\cdot)}/\beta\right)}{\left[1-(1-\gamma)\left(1+\frac{x_{(\cdot)}}{\beta}\right)\exp\left(-x_{(\cdot)}/\beta\right)\right]^2},$$

and

$$\delta_2\left(X_{(\cdot)}\,|\,\mathbf{\Theta}\right) = -\left\{\left[1-\left(1+\frac{x_{(\cdot)}}{\beta}\right)\exp\left(-x_{(\cdot)}/\beta\right)\right] \times \left(1+\frac{x_{(\cdot)}}{\beta}\right)\exp\left(-x_{(\cdot)}/\beta\right)\right\} / \left[1-(1-\gamma)\left(1+\frac{x_{(\cdot)}}{\beta}\right)\exp\left(-x_{(\cdot)}/\beta\right)\right]^2.$$

The $\hat{\mathbf{\Theta}} = (\hat{\gamma}, \hat{\beta})^T$ in Equation (11) and (12) are obtained by numerically solving the system of non-linear equations through the optim function in stats package of R language [13].

3. Results

To illustrate the usefulness of the MADE, we use this method to estimate the unknown parameters of three uncensored real datasets modelled as MOLBE distribution. To compare the MOLBE distribution using the MADE, some other competitive extensions of LBE distribution as well as MO distribution are considered, including the LBE distribution [3], the Marshall-Olkin extended exponential (MOEE) distribution (see [14]), the Weibull length-biased exponential (WLBE) distribution [15] and the Gompertz length-biased (GoLBE) distribution [1]. Their pdfs are expressed by Equation (13)–(16).

The pdf of the LBE distribution is

$$f_{LBE} = \frac{x}{\beta^2}\exp\left(-\frac{x}{\beta}\right), \quad (13)$$

where $x > 0, \beta, \gamma > 0$.

The pdf of the MOEE distribution is

$$f_{MOEE} = \frac{\gamma\beta\exp(-\beta x)}{(1-(1-\gamma)\exp(-\beta x))^2}, \quad (14)$$

where $x > 0, \beta, \gamma > 0$.

The pdf of the WLBE distribution is

$$f_{WLBE} = \alpha\theta\beta^{-2}x\exp\left(-\alpha\left(\frac{1-\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right)}{\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right)}\right)^{\theta}\right) \times\exp\left(-\frac{x}{\beta}\right)\left[\frac{\left(1-\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right)\right)^{\theta-1}}{\left(\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right)\right)^{\theta+1}}\right], \quad (15)$$

where $x > 0, \beta, \gamma, \theta > 0.$

The pdf of the GoLBE distribution is

$$f_{GoLBE} = \theta \frac{x}{\beta^2} \exp\left(-\frac{x}{\beta}\right)\left[\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right)\right]^{-\gamma-1} \times \exp\left(\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right)\right]^{-\gamma}\right\}\right), \quad (16)$$

where $x > 0, \beta, \gamma, \theta > 0.$

The parameters of the corresponding models are estimated by the MADE. Minus log-likelihood (-ll), Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are provided. The values of Anderson–Darling (A-D) statistic and Kolmogorov–Smirnov (K-S) statistic, with its corresponding *p*-value are also obtained. The values of these goodness-of-fit measures are used to indicate the best fitted model. The smaller -ll, AIC, BIC, K-S and A-D values and the higher p-values of K-S and A-D indicate the better fit. For all datasets, the essential computation and data visualization are analyzed by the R language [13].

3.1 Fracture toughness of alumina (Data I)

The first dataset reported by Nadarajah and Kotz [16] represents 119 observations of the fracture toughness of alumina (Al_2O_3) measured in MPa $m^{1/2}$. It has previously been analyzed by Makubate *et al.* [17] and Chipepa *et al.* [18] so as to fit models. Data I is 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

3.2 Breaking stress of carbon fibers (Data II)

The second dataset, reported by Nichols and Padgetteit [19], is the measurement of the tensile strength (GPa) of 66 carbon fibers tested under tension at gauge lengths of 50 mm. Data II is given by: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.90, 1.80, 2.53, 2.88, 3.27, 4.20, 1.84, 2.55, 2.93, 3.28, 4.38, 1.87, 2.55, 2.95, 3.31, 4.42, 1.89, 2.56, 2.96, 3.31, 4.70.

3.3 Annual maximum precipitation (Data III)

The last dataset contains 100 observations of annual maximum precipitation (inches) from 1900 to 1999 for one rain gauge in Fort Collins, Colorado [20]. ZeinEldin *et al.* [21] applied this data to show a high potential of their proposed distribution. The dataset is shown as: 239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223,

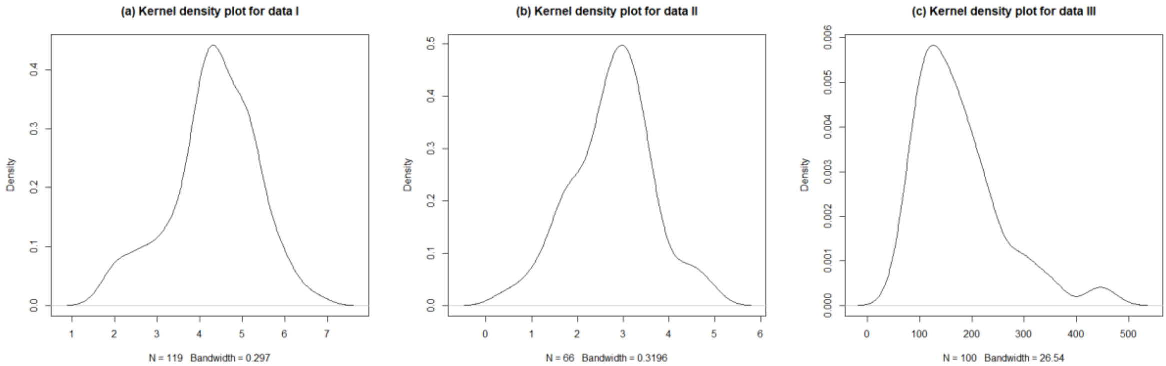

Figure 2 Kernel density plots for three datasets.

215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241.

Table 1 and Figure 2 show the results of the first basic data analysis and kernel density plots for each dataset. The estimated parameters using the MADE (ADEs) and the corresponding goodness-of-fit measure for all of the competitive models are reported in Table 2. More visual comparisons, including histogram with the relative fitted densities (PDFs) and empirical with estimated distribution functions (CDFs), are shown in Figure 2 for the five models.

Upon closer examination of the summary statistics and primary behavior of three datasets reported in Table 1 and Figure 2, we see that Data I, Data II and Data III are approximately slightly left skewed with a moderate kurtosis, symmetrical with a moderate kurtosis, and right skewed with a high kurtosis, respectively. The results presented in Table 2 for Data I and Data II reveals that the MOLBE model has the lowest values for -ll, AIC, BIC and the highest *p*-values (K-S and A-D). Therefore, the MOLBE model provides the best fits among considered distributions. For Data III, the MOLBE model has the smallest values of -ll, AIC as well as BIC with the biggest p-value of A-D but p-value of K-S is a little less than the MOEE model. Nonetheless, the MOLBE should be chosen over the other competing models. Figure 3 also confirms that MOLBE distribution using MADE outperforms all other distributions for these datasets.

Table 1 Descriptive statistics for all of the datasets

Data	Field	n	Min	Mean	Median	SD	Skewness	Kurtosis	Max
I	engineering	119	1.68	4.33	4.38	1.02	–0.42	3.09	6.81
II	engineering	66	0.39	2.76	2.84	0.89	–0.13	3.22	4.90
III	environment	100	60.00	175.67	158.00	83.17	1.34	4.81	463.00

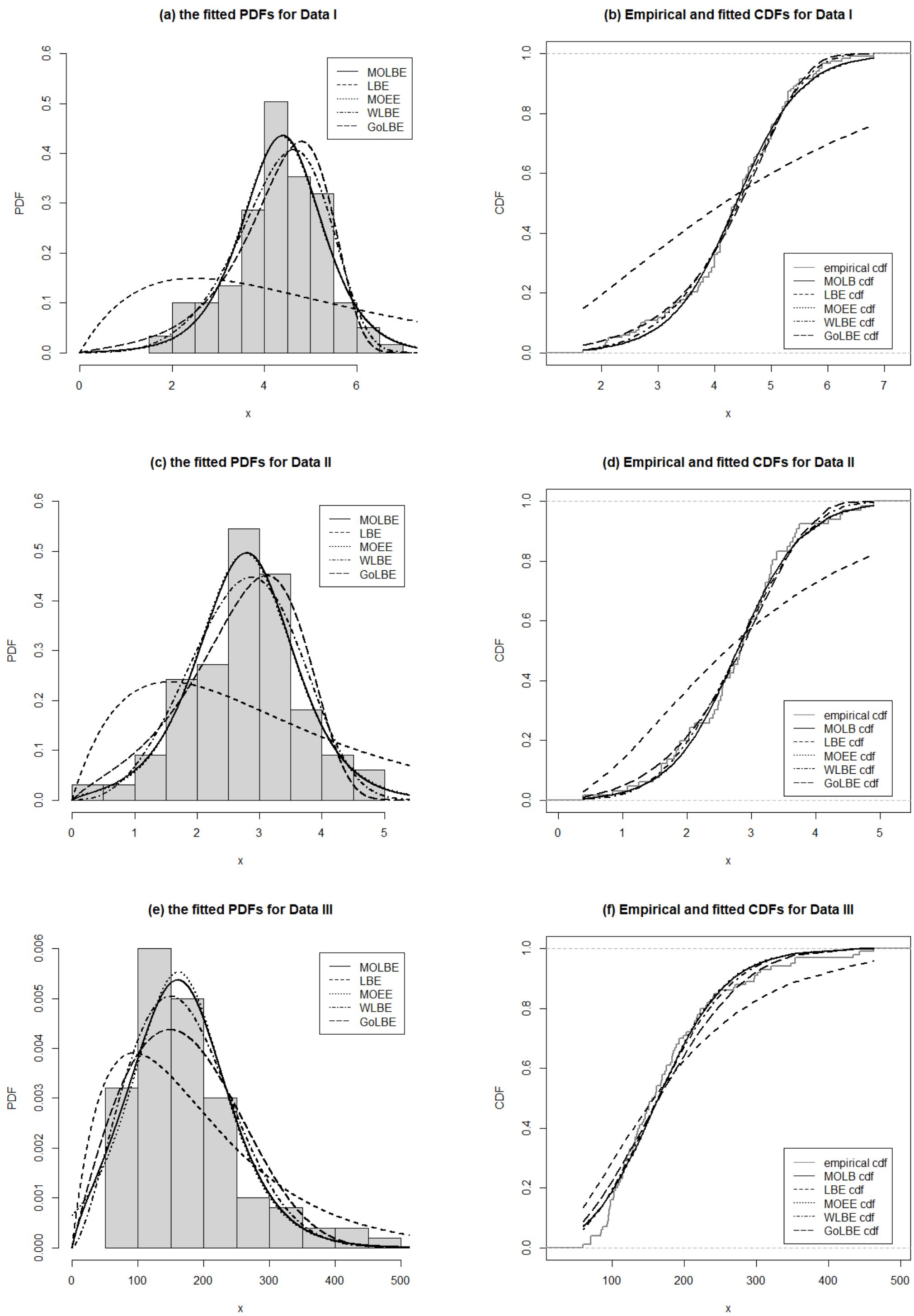

Figure 3 The histograms with PDFs plots (left panels), and the empirical with CDFs plots (right panels) for Data I, Data II and Data III.

Table 2 The ADEs and goodness-of-fit measures for the MOLBE distribution with its competitors

Dataset	Models	Estimates			Statistics						
		β	γ	θ	-ll	AIC	BIC	KS	*p*-value	AD	*p*-value
I	MOLBE	0.515	519.576	-	0.528	5.057	10.615	0.055	0.859	0.528	0.718
	LB	2.473	-	-	19.278	40.555	43.334	0.278	<0.001	19.278	<0.001
	MOEE	1.736	1988.478	-	0.596	5.192	10.750	0.056	0.843	0.596	0.651
	WLB	0.487	0.010	0.625	0.666	7.332	15.670	0.064	0.721	0.666	0.587
	GoLB	1.721	2.635	0.059	0.673	7.346	15.683	0.065	0.697	0.673	0.582
II	MOLBE	0.441	74.117	-	0.305	4.611	8.990	0.062	0.963	0.305	0.934
	LB	1.553	-	-	7.020	16.040	18.230	0.230	0.002	7.020	<0.001
	MOEE	1.973	238.358	-	0.326	4.652	9.031	0.063	0.959	0.326	0.917
	WLB	2.429	2.865	1.860	0.452	6.904	13.473	0.075	0.849	0.452	0.796
	GoLB	0.944	1.391	0.115	0.743	7.487	14.056	0.085	0.721	0.743	0.523
III	MOLBE	41.047	9.959	-	1.507	7.013	12.224	0.094	0.340	1.507	0.175
	LB	94.209	-	-	5.586	13.172	15.777	0.190	0.002	5.586	0.002
	MOEE	0.021	32.854	-	1.534	7.068	12.278	0.093	0.350	1.534	0.169
	WLB	487.485	32.142	1.257	1.529	9.059	16.874	0.096	0.315	1.530	0.170
	GoLB	656.155	7.641	22.847	2.388	10.776	18.592	0.125	0.086	2.388	0.057

4. Discussion and Conclusions

In this research paper, one of the minimum distance estimators, called the MADE, is applied to estimate two unknown parameters of the MOLBE distribution. The MADE can be executed based on approximated cumulative distribution function [8]. To demonstrate the usefulness of the MADE for the MOLBE parameter estimation, three datasets from three different fields each with different characteristics are used in application study. As shown by three applications, the MADE works better for the MOLBE distribution than the others. The MADE has a very good performance in the case of many location scale distributions [8]. Our results indicate that it could be a suitable parameter estimation method for the MOLBE distribution with scale parameter and shape parameter as well. In addition, the -ll, AIC and BIC values of the MOLBE distribution become higher when data are more right skewed. In conclusion, the MOLBE distribution using MADE could be an alternative for practitioners and applied statisticians to model much wider application fields as well as more various types of data. We anticipate that it will grow in popularity in the near future.

References

[1] O. Maxwell, S. O. Oyamakin, and E. J. Thomas, "The gompertz length biased exponential distribution and its application to uncensored data," *Curr Tre Biosta & Biometr*, vol. 1, no. 3, pp. 52–57, 2019.

[2] O. Maxwell, A. I. Friday, and N. C. Chukwudike, "A theoretical analysis of the odd generalized

exponentiated inverse Lomax distribution," *Biometrics & Biostatistics International Journal*, vol. 8, no. 1, pp. 17–22, 2019.

[3] S. T. Dara and M. Ahmad, *Recent Advances in Moment Distribution and their Hazard Rates*. LAP LAMBERT Academic Publishing, 2012.

[4] R. A. Fisher, "The effect of methods of ascertainment upon the estimation of frequencies," *Annals of Eugenics*, vol. 6, no. 1, pp. 13–25, 1934.

[5] M. A. ul Haq, R. M. Usman, S. Hashmi, and A. I. Al-Omeri, "The Marshall-Olkin length-biased exponential distribution and its applications," *Journal of King Saud University - Science*, vol. 31, no. 2, pp. 246–251, 2019.

[6] Ç. Çetinkaya, "Optimal step stress accelerated life testing for the length-biased exponential cumulative exposure model," *İSTATİSTİK*, vol. 13, no. 1, pp. 1–11, 2021.

[7] A. W. Marshall and I. Olkin, "A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families," *Biometrika*, vol. 84, no. 3, pp. 641–652, 1997.

[8] M. Raschke, "Opportunities of the minimum Anderson–Darling estimator as a variant of the maximum likelihood method," *Communications in Statistics - Simulation and Computation*, vol. 46, no. 9, pp. 6879–6888, 2017.

[9] S. Dey, J. Mazucheli, and S. Nadarajah, "Kumaraswamy distribution: Different methods of estimation," *Computational and Applied Mathematics*, vol. 37, no. 2, pp. 2094–2111, 2018.

[10] R. A. ZeinEldin, C. Chesneau, F. Jamal, and M. Elgarhy, "Different estimation methods for type I half-logistic Topp–Leone distribution," *Mathematics*, vol. 7, no. 10, p. 985, 2019.

[11] H. Al-Mofleh, A. Z. Afify, and N. A. Ibrahim, "A new extended two-parameter distribution: Properties, estimation methods, and applications in medicine and geology," *Mathematics*, vol. 8, no. 9, p. 985, 2020.

[12] D.D. Boos, "Minimum Anderson-Darling estimation," *Ommunications in Statistics - Theory and Methods*, vol. 11, no. 24, pp. 2747–2774, 1982.

[13] R Core Team. (2021). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. [Online]. Available: https://www.R-project.org/.

[14] R. Kr. Singh, A. S. Yadav, S. K. Singh, and U. Singh, "Marshall-Olkin extended exponential distribution : different method of estimations," *Journal of Advanced Computing*, vol. 5, no. 1, pp. 12–28, 2016.

[15] N. Ekhosuehi, G. E. Kenneth, and U. K. Kevin, "The Weibull length biased exponential distribution: Statistical properties and applications," *Journal of Statistical and Econometric Methods*, vol. 9, no. 4, pp. 15–30, 2020.

[16] S. Nadarajah and S. Kotz, "Strength modeling using Weibull distributions," *Journal of Mechanical Science and Technology*, vol. 22, no. 7, pp. 1247–1254, 2008.

[17] B. Makubate, F. Chipepa, B. Oluyede, and P.O. Peter, "The Marshall-Olkin half logistic-G family of distributions with applications,"

International Journal of Statistics and Probability, vol. 10, no. 2, pp. 119–136, 2021.
[18] F. Chipepa , B. Oluyede, and P.O. Peter, "The Burr III-Topp-Leone-G family of distributions with applications," *Heliyon*, vol. 7, no. 4, 2021.
[19] M. D. Nicholas and W. J. Padgett, "A bootstrap control chart for Weibull percentiles," *Quality and Reliability Engineering International*, vol. 22, no. 2, pp. 141–151, 2005.
[20] M. M. Ristic, and N. Balakrishnan, "The gamma-exponentiated exponential distribution," *Journal of Statistical Computation and Simulation*, vol. 82, no. 8, pp. 1191–1206, 2012.
[21] R. A. ZeinEldin, F. Jamal, C. Chesneau, and M. Elgarhy, "Type II Topp–Leone inverted kumaraswamy distribution with statistical," *Symmetry*, vol. 11, no. 12, pp. 1191–1206, 2019.