Research Article

The Performance of CUSUM Control Chart for Monitoring Process Mean for Autoregressive Moving Average with Exogenous Variable Model

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Received: 14 April 2020; Revised: 26 August 2020; Accepted: 10 September 2020; Published online: 23 November 2020

Abstract
The objective of this study was to derive explicit formulas for the average run length (ARL) of an autoregressive moving average with an exogenous variable (ARMAX(p,q,r)) process with exponential white noise on a cumulative sum (CUSUM) control chart. To check the accuracy of the ARL derivations, the efficiency of the proposed explicit formulas was compared with a numerical integral equation (NIE) method in terms of the absolute percentage error. There was excellent agreement between the two methods, but when comparing their computational times, the explicit formulas only required 1 second whereas the NIE method required 599.499–835.891 s. In addition, real-world application of the derived explicit formulas was illustrated using Hong Kong dollar exchange rates data with an exogenous variable (the US dollar) to evaluate the ARL of an ARMAX (p,q,r) process on a CUSUM control chart.

Keywords: Explicit formulas, Average Run Length (ARL), ARMAX(p,q,r)) process, Exponential white noise

1 Introduction

The control chart is an essential tool for statistical process control (SPC) that is widely used to monitor the quality and to improve a process. Walter A. Shewhart pioneered the concept of the control chart in the 1920s, with the original application being primarily for manufacturing processes. Afterward, control charts have been used for development and improvement of processes, which, in a practical sense, has been extended for use in many applications, including healthcare and public-health surveillance, analytical laboratories, nuclear power plant control rooms, monitoring of coal quality, etc. Three control chart schemes have been developed for SPC, namely Shewhart [1], cumulative sum (CUSUM) [2], and exponentially weighted moving average (EWMA) [3]. Although the Shewhart control chart performs well for detecting large process changes in the mean, it is inefficient for detecting small-to-moderate ones.

The performances of the CUSUM and EWMA control charts are similar. Both are good alternatives to the Shewhart control chart when one is interested in the detection of small-to-moderate changes in the process mean. The CUSUM control chart is widely used in the manufacturing industry to detect changes in the quality of manufactured products and its application has been widely proposed in the statistical literature (e.g. [4]). It has also been used in healthcare, such as monitoring HIV/AIDS patients in Nigeria [5]. Besides, Van Dobben de Bruyn [6] presented a general description of the construction of one-sided and two-sided CUSUM control charts.

The use of CUSUM control charts for monitoring autocorrelated processes has been studied in many aspects, such as for monitoring the process mean for the situation in which observations follow a 1st order autoregressive (AR(1)) process with additional random errors [7]. Moreover, a distribution-free tabular CUSUM chart for autocorrelated data was purposed by Kim *et al*. [8], while Chang and Wu [9] studied the run length properties of Shewhart, CUSUM, and EWMA charts in a unified functional Markov chain embedding approach for AR(1) and AR(2) processes.

In statistical time series analysis, the error term "white noise" usually follows a normal distribution. However, in practical problems, correlated observations can occur in some processes when the errors comprise exponential white noise. Moreover, these correlations can affect the properties of CUSUM charts. For instance, Jacob and Lewis [10] considered an autoregressive moving average process with order (1,1) (ARMA(1,1)) when observations are exponentially distributed with exponential white noise. Later, a Bayesian analysis of an autoregressive model with order 1 (AR(1)) following an exponential distribution was conducted by Mohamed and Hocine [11], while Pereira and Turkrman [12] used exponential white noise in a Bayesian analysis of threshold AR models. Recently, Suparman [13] estimated the parameters of an AR model with exponential white noise when the order was unknown.

The performance of a CUSUM control chart is evaluated in terms of the average run length (ARL), which is the most used statistic for measuring the performance of quality control charts, including the first alarm. ARL0 is while there is no change in the statistical process (i.e. in-control) and ARL1 is when the process first becomes out-of-control. Therefore, the ARL value needs to be as large as possible when the process is in control and as small as possible when the process is out of control. In general, it is relatively straightforward to calculate these values and has been used as an argument in discussions by researchers in a wide variety of applications. For instance, the properties of the ARL for a one-sided CUSUM chart [4], [14], [15] and calculating the ARL under time series models such as AR(1), MA(1), and ARMA(1,1) when the time series is stationary [16], [17].

Based on the researches previously mentioned, there are various methods for the calculation of the ARL for the control chart. Page [2] initially derived integral equations for the ARLs of a one-sided CUSUM control chart, while Goel and Wu [18] obtained approximate ARLs in the normal case by employing ratios of numerical solutions with two integral equations. The Markov chain approach (MCA) proposed by Brook and Evans [14] was used to study the run length properties of CUSUM charts based on the assumption of independent and identically distributed (i.i.d) observations. Using MCA to obtain the approximate ARLs for the exponential case was employed by Lucas [19] while solving the Page's [2] integral equation to obtain the ARLs for the exponential case was solved by Vardeman and Ray [20]. The performance of CUSUM control charts for monitoring the process mean was evaluated via a simulation approach by Jun and Choi [21] in 1993. Deriving a numerical integral equation (NIE) to determine the ARLs of control charts was employed by Crowder [22]; he used the Fredholm integral equation of the second kind for approximations of the run length and its variance using a system of linear equations to obtain exact expressions for the mean and variance.

These methods are often used to determine the performance of the ARL of control charts. However, they are fairly difficult and complex calculations. Consequently, a relatively new approach has been to derive explicit formulas that are good alternatives. For instance, in the explicit formulas for the ARL proposed by Mititelu *et al*. [23], they used the Fredholm integral equation for a one-sided EWMA control chart with a Laplace distribution and a CUSUM control chart with a hyper exponential distribution. Recently, Petcharatet *et al*. [24] derived explicit formulas for the ARLs of a CUSUM chart in cases of MA(q) with exponential white noise using integral equations based on the Fredholm integral equation of the second kind. Later, an analytical solution for the ARL of a CUSUM control chart for an autoregressive process with one explanatory variable (ARX(1)) with exponential white noise was presented by Paichit [25]. Finally, Peerajit *et al*. [26] also studied that observations are long memory processes with non-seasonal and seasonal ARFIMA model with exponential white noise when the NIE method is applied for ARL approximation on CUSUM chart.

In this paper, the explicit formulas for the ARL of a CUSUM control chart for an ARMA with an exogenous variable (ARMAX(p,q,r)) process with exponential white noise are derived. The rest of this paper is organized as follows. The characteristics of the CUSUM control chart and ARMAX(p,q,r) processes along with the assumptions for calculating the ARL are presented in Section 2. The explicit solution for the ARL of an ARMAX(p,q,r) process is offered in Section 3. The integral equation to guarantee the existence and uniqueness of the solution for calculating the ARL via explicit formulas using Banach's fixed point theorem is proven in Section 4. The numerical scheme to evaluate the solution of the integral equations (i.e. the NIE) is presented. The results of analytical study to compare ARL0 and ARL1 of an ARMAX(p,q,r) process on a CUSUM control chart are reported in Section 5. In Section 6, real Hong Kong dollar exchange rate data with an exogenous variable (the US dollar) are used to evaluate the ARLs derived from explicit formulas and the NIE method. Last, conclusions are provided in Section 7.

2 The CUSUM Control Chart for ARMAX(p,q,r) Process and Characteristics

The CUSUM control chart can be defined as follows [Equation (1)] [2]:

$$Z_t = \max(Z_{t-1} + Y_t - a, 0), \text{ t} = 1,2,... \tag{1}$$

where Z_t is the CUSUM statistic, Y_t is the sequence of an ARMAX(p,q,r) process with exponential white noise, $Z_0 = s$ is an initial value, and a is the constant recall reference value for the chart.

The ARMAX(p,q,r) process is described by the following recursion [Equation (2)]:

$$Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - ... - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\omega_i X_{it} \;\; ; t = 1,2,... \tag{2}$$

where ε_t are independent and identically distributed (i.i.d.) observations in an exponential distribution, X_t are exogenous variables, and are coefficients. The initial value $\varepsilon_0 = 1$ for the AR coefficient $-1 \le \phi_i \le 1$ and the MA coefficient $-1 \le \theta_i \le 1$. It is assumed that the initial value for the ARMAX(p,q,r) process is 1.

In this paper, we consider SPC charts under the assumption that sequential observations $\varepsilon_1, \varepsilon_2, ...,$ are independent random variables with distribution function $F(x, \lambda)$ The parameter $\lambda = \lambda_0$ before the change-point time ($\theta \le \infty$) In the in-control state, $\theta = \infty$ means that the change-point time has not yet been reached. However, $\lambda > \lambda_0$ when the change-point time (θ) is reached (the out-of-control state). The typical condition on the choice of stopping time τ is as follows:

$$\text{ARL}_0 = \mathbf{E}_\theta(\tau_h) \tag{3}$$

where $\mathbf{E}_\infty(.)$ denotes the expectation under distribution $F(x, \lambda_0)$ (in-control) that the change-point occurs at point θ, ($\theta \le \infty$). In the literature on quality control, quantity $\mathbf{E}_\infty(\tau)$ the ARL of the in-control process in Equation (3). Subsequently, by definition, $\text{ARL}_0 = \mathbf{E}_\infty(\tau)$. and a typical practical constraint is $\text{ARL}_0 = T$.

Another typical constraint consists of minimizing the quantity [Equation (4)]:

$$\text{ARL}_1 = \mathbf{E}_\theta(\tau_h | \tau_h \ge 1), \tag{4}$$

where $\mathbf{E}_\infty(.)$ is the expectation under distribution $F(x, \lambda)$ (out-of-control) and λ_1 is the value of a parameter after the change-point.

The first passage of time on the CUSUM control chart is given by [Equation (5)]

$$\tau_h = \inf\{t > 0 : Z_t > h\}, \tag{5}$$

where h is a known constant parameter for the upper control limit.

3 ARL Explicit Formulas for an ARMAX(p,q,r) Process on a CUSUM Control Chart

In this section, the explicit solution for the ARL using the Fredholm integral equation of the second kind is derived. Let $C(s) = \mathbf{E}(\tau_h) < \infty$ be the ARL of the CUSUM control chart after it has been reset to $s \in [0, h]$. The solution for the integral equation is as follows:

$$C(s) = 1 + \mathbf{E}_z[I\{0 < Z_1 < h\}C(Z_1)] + \mathbf{P}_z\{Z_1 = 0\}\, C(0).$$

Theorem 3.1 The explicit formulas for the ARL of an ARMAX(p,q,r) process on a CUSUM control chart is

$$C(s) = e^{\lambda h}\left(1 + e^{\lambda\left(\begin{matrix} a-\mu-\phi_1 Y_{t-1}-\phi_2 Y_{t-2}-...-\phi_p Y_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_i X_{it}\end{matrix}\right)} - \lambda h\right) - e^{\lambda s},$$

where $s \ge 0$.

***Proof*.**

$$C(s)=1+\int_0^h C(y)\lambda e^{\lambda\left(\begin{matrix} s-a+\mu+\phi_1Y_{t-1}+\phi_2Y_{t-2}+...+\phi_pY_{t-p} \\ -\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}e^{-\lambda y}dy$$

$$+(1-e^{-\lambda\left(\begin{matrix} a-s-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)})C(0);\ s\in[0,a) \quad (6)$$

Let g be constant as $g=\int_0^h C(y)e^{-\lambda y}dy$. $C(s)$ can be written as

$$C(s)=1+\lambda g e^{\lambda\left(\begin{matrix} s-a+\mu+\phi_1Y_{t-1}+\phi_2Y_{t-2}+...+\phi_pY_{t-p} \\ -\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}$$

$$+(1-e^{-\lambda\left(\begin{matrix} a-s-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)})C(0)$$

For s = 0 then

$$C(0)=1+\lambda g e^{\lambda\left(\begin{matrix} -a+\mu+\phi_1Y_{t-1}+\phi_2Y_{t-2}+...+\phi_pY_{t-p} \\ -\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}$$

$$+(1-e^{-\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)})C(0)$$

$$=e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}+\lambda g$$

Then

$$C(s)=1+\lambda g e^{\lambda\left(\begin{matrix} s-a+\mu+\phi_1Y_{t-1}+\phi_2Y_{t-2}+...+\phi_pY_{t-p} \\ -\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}$$

$$+1-e^{-\lambda\left(\begin{matrix} a-s-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}$$

$$\times e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}+\lambda g$$

$$=1+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}+\lambda g-e^{\lambda s}$$

$$=1+\lambda g+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}-e^{\lambda s}. \quad (7)$$

Now, constant g can be found:

$$g=\int_0^h C(y)e^{-\lambda y}dy$$

$$=\int_0^h (1+\lambda g+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}-e^{\lambda y})e^{-\lambda y}dy$$

$$=\int_0^h (1+\lambda g+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)})e^{-\lambda y}dy-\int_0^h e^{\lambda y}.e^{-\lambda y}dy$$

$$=(1+\lambda g+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}).(\frac{1}{\lambda}(1-e^{-\lambda h}))-h$$

Multiplying both sides of the equation by λ

$$\lambda g=(1+\lambda g+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}).(1-e^{-\lambda h})-\lambda h$$

$$=1+\lambda g+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}-e^{-\lambda h}-\lambda g e^{-\lambda h}$$

$$-e^{\left(\begin{matrix} \lambda a-\lambda\mu-\lambda\phi_1Y_{t-1}-\lambda\phi_2Y_{t-2}-...-\lambda\phi_pY_{t-p} \\ +\lambda\theta_1\varepsilon_{t-1}+\lambda\theta_2\varepsilon_{t-2}+...+\lambda\theta_q\varepsilon_{t-q}-\lambda\sum_{i=1}^{r}\omega_iX_{it}-\lambda h\end{matrix}\right)}-\lambda h$$

$$=(1-e^{-\lambda h})+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}.(1-e^{-\lambda h})$$

$$+\lambda g(1-e^{-\lambda h})-\lambda h$$

$$\lambda g(e^{-\lambda h})=(1-e^{-\lambda h})(1+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)})-\lambda h$$

$$g=\frac{(1-e^{-\lambda h})}{\lambda e^{-\lambda h}}(1+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)})-\frac{h}{e^{-\lambda h}}$$

$$\therefore\ g=\frac{e^{\lambda h}}{\lambda}(1-e^{-\lambda h})(1+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)})-he^{\lambda h}$$

Substituting constant g into Equation (7), we obtain

$$C(s)=e^{\lambda h}\left(1+e^{\lambda\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}-\lambda h\right)-e^{\lambda s},\ \ s\geq 0.$$

As previously mentioned, $\lambda=\lambda_0$ implies that the process is in the in-control state. Thus, the explicit analytical solution for ARL_0 can be written as

$$\mathrm{ARL}_0=e^{\lambda_0 h}(1+e^{\lambda_0\left(\begin{matrix} a-\mu-\phi_1Y_{t-1}-\phi_2Y_{t-2}-...-\phi_pY_{t-p} \\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_iX_{it}\end{matrix}\right)}-\lambda_0 h)-e^{\lambda_0 s}. \quad (8)$$

On the contrary, if the process is in the out-of-control state, the value of exponential parameter $\lambda = \lambda_0(1 + \delta)$, where $\lambda > \lambda_0$ and δ is the shift size. Hence, the explicit analytical solution for ARL_1 can be expressed as [Equation (9)]

$$\text{ARL}_1 = e^{\lambda_1 h}\left(1+e^{\lambda_1\left(\begin{array}{l}a-\mu-\phi_1 Y_{t-1}-\phi_2 Y_{t-2}-\ldots-\phi_p Y_{t-p}\\ +\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\omega_i X_{it}\end{array}\right)}-\lambda_1 h\right)-e^{\lambda_1 s}. \quad (9)$$

4 The Existence and Uniqueness of the Solution for the ARL Using the Explicit Formulas

The ARL is often computed as a numerical solution to an integral equation. In this section, some important definitions from real analysis and Banach's fixed point theorem to verify the existence and uniqueness of the explicit analytical solution for ARL are derived.

Definition 4.1: Sequence $\{u_n\}_{n\geq 0}$ in (M, d) is said to converge if $u \in M$ such that $N \in Z$ exists for all $\varepsilon > 0$ and $d(u_n, u) < \varepsilon$ exists for all $n \geq N$.
Definition 4.2: Sequence $\{u_n\}_{n\geq 0} \in M$ is said to be Cauchy if $N \in Z$ exists for all $\varepsilon > 0$ and $d(u_m, u_n) < \varepsilon$ exists for all $m, n \geq N$.
Definition 4.3: The metric space is said to be complete if all Cauchy sequences $\{u_n\}_{n\geq 0}$ converge.
Definition 4.4: The fixed point of continuous function T; $M \to M$ is point for which u^*, $T(u^*) = u^*$.
Definition 4.5: Let (M, d) be a metric space. Function T; $M \to M$ is said to be a contraction if $0 \leq \eta < 1$ such that $d(T(u), T(v)) \leq \eta d(u, v)$ for all $u, v \in X$.
Note that: Metric space (M, d) is vector space M with metric d.
Theorem 4.1: (Banach's fixed point theorem) [27].
Let (M, d) be a complete metric space. Suppose T; $M \to M$ is a contraction, then T is a specific and unique fixed point.
Remark: Metric space (M, d) is vector space M with metric d. Thus, normed vector space $(M, \|.\|)$ comprising vector space M with norm $\|.\|$ can be defined by $d(u, v) = \|u - v\|$.
Theorem 4.2: The ARL of an ARMAX(p,q,r) process on a CUSUM control chart when assuming the existence and uniqueness of the explicit analytical solution for ARL can be obtained by solving the integral equation.
Proof: [Existence].
Let T be a contraction in complete metric space (M, d), then arbitrarily take $\text{ARL}_1 \in M$ and define sequence $\{\text{ARL}_n\}_{n\geq 0}$ by setting $\text{ARL}_n = \text{T}(\text{ARL}_{n-1})$ for each $n \geq 1$. First of all, via iteration we can obtain

$$\begin{aligned} d(\text{ARL}_{n+1}, \text{ARL}_n) &= d(T(\text{ARL}_n), T(\text{ARL}_{n-1})) \\ &\leq \eta d(\text{ARL}_n, \text{ARL}_{n-1}) \end{aligned}$$

where
$\eta d(\text{ARL}_n, \text{ARL}_{n-1}) = \eta d(T(\text{ARL}_{n-1}), T(\text{ARL}_{n-2}))$
$d(\text{ARL}_{n+1}, \text{ARL}_n) = \eta^n d(\text{ARL}_1, \text{ARL}_0)$; for each $n > 0$.

The distance between ARL_m and ARL_n, for all $m > n \geq N$, can then be estimated using the triangle inequality.

$$\begin{aligned} d(\text{ARL}_m, \text{ARL}_n) &\leq \sum_{i=n}^{m-1} d(\text{ARL}_{i+1}, \text{ARL}_i) \\ &\leq \sum_{i=n}^{m-1} \eta^i d(\text{ARL}_1, \text{ARL}_0). \end{aligned}$$

Using the formula for the sum of geometric series, we obtain

$$d(\text{ARL}_m, \text{ARL}_n) \leq \frac{\eta^n}{1-\eta} d(\text{ARL}_1, \text{ARL}_0).$$

Since $0 \leq \eta < 1$ implies that $d(\text{ARL}_m, \text{ARL}_n) \to 0$ as $n \to \infty$. That is to say, $\{\text{ARL}_n\}_{n\geq 0}$ is a Cauchy sequence. As (M, d) is complete, $\{\text{ARL}_n\}$ converges to $\text{ARL} \in M$. Thus, there exists a unique point $\text{ARL}_0 \in M$ such that

$$\begin{aligned} T(\text{ARL}) = T(\lim_{n\to\infty} \text{ARL}_n) &= \lim_{n\to\infty} T(\text{ARL}_n) \\ &= \lim_{n\to\infty} \text{ARL}_{n+1} = \text{ARL}, \end{aligned}$$

where ARL is the fixed point of T.
Proof: [Uniqueness].
To show that operator T is the contraction mapping, let ARL_1 and ARL_2 be two arbitrary functions in $\mathbf{C}[0, h]$ Assume that the pair $(M, \| \|)_\infty$ is the complete metric space where the particular metric $M = \mathbf{C}[0, h]$ is a set of all continuous functions of ARL defined on[0, h], then $\mathbf{C}([0, h])$ becomes norm space if we define

$$\| \text{ARL} \|_\infty = \sup_{s\in[0,h]} \left| \int_0^h k(s, y)dy \right|,$$

for every function $\text{ARL}(.) \in \mathbf{C}([0, h])$ where $k(s, y)$ is called the kernel function of the integral equation for ARL obtained by using Equation (6):

$$k(s, y) = \lambda e^{\lambda\left(\begin{array}{l}s-a-y+\mu+\phi_1 Y_{t-1}+\phi_2 Y_{t-2}+\ldots+\phi_p Y_{t-p}\\ -\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\omega_i X_{it}\end{array}\right)}$$

$$\|T(\text{ARL}_1)-T(\text{ARL}_2)\|_\infty$$
$$= \sup_{s\in[0,h]}\left|\int_0^h k(s,y)\left|\text{ARL}_1(y)-\text{ARL}_2(y)\right|dy\right|$$
$$\le \sup_{s\in[0,h]}\int_0^h \left|k(s,y)\right|\left|\text{ARL}_1(y)-\text{ARL}_2(y)\right|dy$$
$$\le \sup_{s\in[0,h]}\int_0^h \left|k(s,y)\right|dy\left\|\text{ARL}_1(y)-\text{ARL}_2(y)\right\|_\infty$$
$$= \eta\left\|\text{ARL}_1(y)-\text{ARL}_2(y)\right\|_\infty$$

Therefore, $\|T(\text{ARL}_1)-T(\text{ARL}_2)\|_\infty = \eta\|\text{ARL}_1-\text{ARL}_2\|_\infty$ where $\eta < 1$ and $\eta = \sup_{s\in[0,h]}\int_0^h |k(s,y)|dy$ is a positive constant. The triangular inequality can be used for the supremum norm because

$$\left|\text{ARL}_1(0)-\text{ARL}_2(0)\right| \le \sup_{s\in[0,h]}\left|\text{ARL}_1(s)-\text{ARL}_2(s)\right| = \left\|\text{ARL}_1-\text{ARL}_2\right\|_\infty$$

That is to say, T: $\mathbf{C}[0, h] \to \mathbf{C}[0, h]$ is the contraction mapping in the complete metric space $(\mathbf{C}[0, a], \|.\|)$. By Theorem 4.1, there exists a unique solution such that $T(\text{ARL})(s) = \text{ARL}(s)$.

This completes the proof. Therefore, the explicit formulas for the ARL of an ARMAX(p,q,r) process on a CUSUM control chart have been verified in terms of existence and uniqueness.

5 Numerical Results

In this section, the results for ARL0 and ARL1 of an ARMAX(p,q,r) process on a CUSUM chart are compared. The numerical scheme to evaluate the solution of the integral equations (i.e. the NIE) is given by

$$\text{ARL}_{\text{NIE}}(u) = 1 + \text{ARL}_{\text{NIE}}(a_1)F(a-s-Y_t) + \sum_{j=1}^{m} w_j\, \text{ARL}_{\text{NIE}}(a_j) f(a_j+a-s-Y_t). \quad (10)$$

where $a_j = \frac{h}{m}(j-\frac{1}{2})$ and $w_j = \frac{h}{m}; j = 1,2,...,m.$

The ARL results are reported in Table 1. The parameter values chosen for the CUSUM control chart were a = 2, 2.5, and 3, the desired ARL_0 = 370, in-control parameter λ_0 = 1, and magnitude of the change is λ_1. Using the CUSUM chart parameters a and h (the CUSUM control limit), each model was selected to give the desired in-control ARL_0 = 370. For the parameters h are calculated from Equation (8). It was found that parameter a increased while h decreased in the process. We considered the performance of the proposed explicit formulas in terms of the computational time and the difference between the absolute percentage errors computed as follows:

$$Diff(\%) = \frac{\left|\text{ARL}_{\text{Explicit formulas}} - \text{ARL}_{\text{NIE}}\right|}{\text{ARL}_{\text{Explicit formulas}}} \times 100.$$

We also compared the computational times required to compute the numerical values for ARL_0 and ARL_1.

Note that λ_0 = 1 is the value of the in-control parameter. The columns for $\lambda > 1$ correspond to the out-of-control parameter and are thus the values for ARL_1. The numbers in each cell in the table represent the value of ARL_1 and the computational time for the calculations are in parentheses.

The results reported in Table 1 show that those for the proposed explicit formulas are close to those for NIE. It should be mentioned that the results from both methods in terms of the difference between their absolute percentage errors (*Diff*) were less than 0.35%. The efficiency when calculating ARL_1 depends on the computation time; the explicit formulas required a computational time of less than 1 s. whereas NIE required 599.499–835.891 seconds.

The results comparing the reference values with a = 2, 2.5, and 3 for ARL_0 = 370 are presented in Figures 1–3. Considering the mean changes (λ_1), the magnitude of λ increases from 1 to 5, respectively. It was found that the ARL1 using the explicit formulas decreased for each level in the process mean and every reference value parameter (a) of the CUSUM chart. In addition, a = 2 gave the lowest ARL_1 results for all magnitudes of process shifts for all models and was thus more effective for detecting shifts in the process mean than a = 2.5 and 3.

Table 1: ARL1 values for ARMAX(p,q,r) of CUSUM chart using explicit formulas against NIE at ARL_0 = 370

Models	Parameters	*a*	*h*	ARL	λ_1									
					1.01	1.03	1.05	1.07	1.10	1.30	1.50	2.00	3.00	5.00
ARMAX (1, 2, 1)	ϕ_1 = 0.1, θ_1 = 0.1, θ_2 = –0.2, ω_1 = 0.75	2.0	4.5801	Explicit	344.72	299.774	262.142	230.449	191.734	71.333	35.872	13.104	5.541	1.139
				CPU_{Exp}	0.014	0.014	0.014	0.013	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	343.473	298.739	261.278	229.723	191.17	71.196	35.826	13.079	5.539	1.138
				CPU_{NIE}	631.781	652.688	648.25	627.562	630.453	636.594	765.469	655.281	643.609	645.123
				Diff(%)	0.36	0.35	0.33	0.32	0.29	0.19	0.13	0.19	0.04	0.09
		2.5	3.663	Explicit	347.003	306.268	271.607	241.959	205.069	83.097	43.288	15.599	6.052	2.986
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	345.855	305.283	270.757	241.222	204.470	82.909	43.210	15.583	6.049	2.986
				CPU_{NIE}	637.937	726.032	727.266	729.516	784.359	787.391	704.89	709.156	707.938	710.734
				Diff(%)	0.33	0.32	0.31	0.30	0.29	0.23	100.00	0.10	0.05	0.00
		3.0	3.023	Explicit	348.162	308.976	275.44	246.593	210.462	88.271	46.824	16.999	6.425	3.059
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.015	0.014	0.014	0.014	0.014	0.014
				NIE	347.171	308.117	274.693	245.94	209.924	88.088	46.744	16.98	6.421	3.058
				CPU_{NIE}	835.891	648.775	644.453	690.391	689.515	731.653	705.359	680.234	689.516	692.047
				Diff(%)	0.28	0.28	0.27	0.26	0.26	0.21	0.17	0.11	0.06	0.03
ARMAX (1, 3, 1)	ϕ_1 = –0.1, θ_1 = –0.1, θ_2 = –0.2, θ_3 = –0.3, ω_1 = 0.5	2.0	5.445	Explicit	339.508	286.406	243.283	208.207	166.498	52.055	25.026	10.217	5.185	2.994
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	338.002	286.014	243.189	208.122	166.492	52.071	25.011	10.213	5.176	2.988
				CPU_{NIE}	658.249	609.283	710.014	705.219	678.958	645.253	689.009	701.016	643.129	684.160
				Diff(%)	0.44	0.14	0.04	0.04	0.01	0.03	0.06	0.04	0.17	0.20
		2.5	3.9697	Explicit	346.283	304.468	269.034	238.846	201.462	79.799	41.131	14.818	5.870	2.959
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	345.080	303.443	268.157	238.091	200.854	79.617	41.0588	14.803	5.867	2.959
				CPU_{NIE}	603.328	599.499	611.172	603.063	599.750	600.922	603.703	600.719	616.437	658.719
				Diff(%)	0.35	0.34	0.33	0.32	0.30	0.23	0.18	0.10	0.05	0.00
		3.0	3.265	Explicit	347.839	308.154	274.253	245.143	208.758	86.578	45.642	16.513	6.289	3.030
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	348.001	308.045	274.569	245.098	208.145	86.579	45.528	16.269	6.236	3.695
				CPU_{NIE}	695.269	679.021	615.228	708.208	639.228	699.256	668.429	692.128	696.129	666.173
				Diff(%)	0.05	0.04	0.12	0.02	0.29	0.02	0.25	0.27	0.21	0.10
ARMAX (2, 3, 1)	ϕ_1 = 0.1, ϕ_2 = 0.2, θ_1 = 0.1, θ_2 = 0.2, θ_3 = 0.3, ω_1 = 0.5	2.0	3.9711	Explicit	346.694	304.819	269.336	239.107	201.673	79.863	41.157	14.824	5.872	2.960
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	346.181	304.615	269.235	239.201	201.539	79.855	41.152	14.819	5.856	2.961
				CPU_{NIE}	630.259	689.103	710.269	695.269	659.358	645.293	661.289	682.036	638.069	685.236
				Diff(%)	0.15	0.07	0.04	0.04	0.07	0.01	0.01	0.03	0.27	0.03
		2.5	3.265	Explicit	347.839	308.154	274.253	245.143	208.758	86.5784	45.642	16.513	6.289	3.030
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	346.783	307.243	273.462	244.453	208.191	86.391	45.561	16.494	6.285	3.029
				CPU_{NIE}	714.300	709.110	716.093	686.392	683.687	697.75	702.953	699.828	699.766	712.16
				Diff(%)	0.30	0.30	0.29	0.28	0.27	0.22	0.18	0.12	0.06	0.03
		3.0	2.6812	Explicit	348.621	309.966	276.821	248.06	212.406	90.22	48.213	17.596	6.603	3.100
				CPU_{Exp}	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
				NIE	347.968	309.858	276.751	248.012	212.218	90.568	48.159	17.569	6.596	3.112
				CPU_{NIE}	702.589	698.025	658.215	711.258	689.258	685.256	651.489	689.008	645.189	698.179
				Diff(%)	0.19	0.03	0.03	0.02	0.09	0.39	0.11	0.15	0.11	0.39

CPU_{Exp} and CPU_{NIE} are the computational time (Second) of explicit formulas and NIE method, respectively.

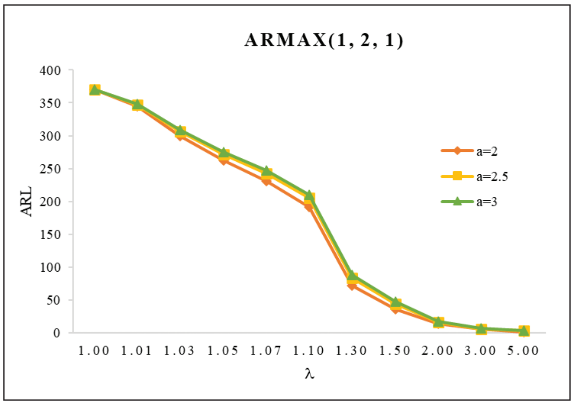

Figure 1: ARL values for ARMAX(1, 2, 1) of CUSUM chart using explicit formulas.

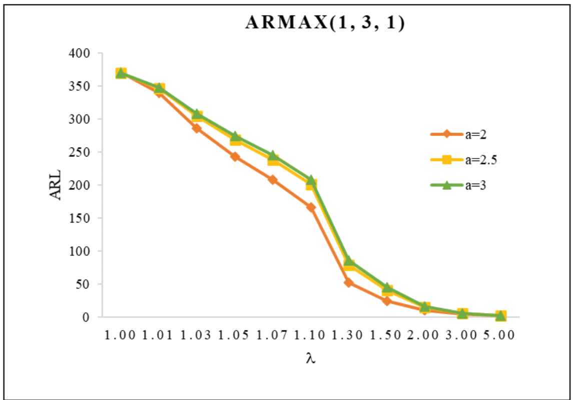

Figure 2: ARL values for ARMAX(1, 3, 1) of CUSUM chart using explicit formulas.

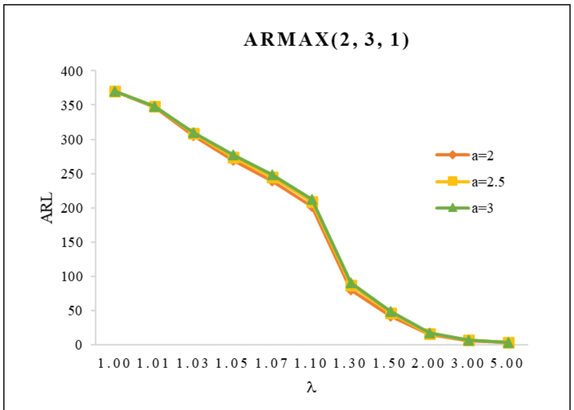

Figure 3: ARL values for ARMAX(2, 3, 1) of CUSUM chart using explicit formulas.

6 Application of the Proposed Explicit Formulas with Real Data

The results of evaluating the ARLs from the explicit formulas and NIE using Hong Kong dollar exchange rate data [28] with an exogenous variable (the US dollar) are reported in Tables 2 and 3.

Table 2: Comparison of ARL value between explicit formulas and NIE method with data on Hong Kong dollar exchange rate for $\phi_1 = 0.311162$, $\phi_2 = 0.618779$, $\theta_1 = 0.99723$, and $\omega_1 = 0.99723$ at $ARL_0 = 370$

δ	$a = 1.45, h = 0.01095$				*Diff*(%)
	Explicit		NIE		
	ARL	CPU	ARL	CPU	
0.000	370.014	0.014	368.761	657.513	0.34
0.001	71.537	0.016	71.382	659.214	0.22
0.002	27.646	0.014	27.605	634.003	0.15
0.003	15.099	0.014	15.083	637.186	0.11
0.004	9.986	0.014	9.978	638.699	0.09
0.005	7.403	0.014	7.398	639.620	0.07
0.006	5.903	0.014	5.899	640.165	0.07
0.007	4.942	0.014	4.940	638.730	0.04
0.008	4.282	0.014	4.281	636.609	0.02
0.009	2.192	0.014	2.192	640.603	0.00
0.100	1.722	0.014	1.722	642.584	0.00

CPU is the computational time (Second) of method.

Table 3: Comparison of ARL value between explicit formulas and NIE method with data on Hong Kong dollar exchange rate for $\phi_1 = 0.311162$, $\phi_2 = 0.618779$, $\theta_1 = 0.99723$, and $\omega_1 = 0.99723$ at $ARL_0 = 500$

δ	$a = 1.45, h = 0.01195$				*Diff*(%)
	Explicit		NIE		
	ARL	CPU	ARL	CPU	
0.000	500.252	0.014	498.385	640.493	0.37
0.001	87.219	0.014	87.016	641.321	0.23
0.002	31.800	0.014	31.750	643.270	0.16
0.003	16.793	0.015	16.774	640.135	0.11
0.004	10.883	0.014	10.873	640.010	0.09
0.005	7.964	0.014	7.959	638.402	0.06
0.006	6.294	0.014	6.291	638.060	0.05
0.007	5.236	0.014	5.234	639.214	0.04
0.008	4.516	0.014	4.514	637.997	0.04
0.009	2.263	0.017	2.263	639.588	0.00
0.100	1.764	0.014	1.764	639.433	0.01

CPU is the computational time (Second) of method.

The observations were collected monthly from January 2015 to July 2019, and the dataset was shown to follow an ARMAX(2, 1, 1) process with coefficients ϕ_1 = 0.311162, ϕ_2 = 0.618779, θ_1 = 0.997230, ω_1 = 0.997230, and the errors were exponential white noise with parameter λ = 0.00295. For the CUSUM control chart, reference value a = 1.45 and control limit value h = 0.01095, and 0.01195 were used for ARL_0 = 370 and 500, respectively. The results in Tables 2 to 3 follow a similar trend to those in Table 1. The numerical results from the proposed formulas are very close to the NIE results for all shift sizes in the process mean.

7 Conclusions

Explicit formulas for ARL of an ARMAX(p,q,r) process with exponential white noise on a CUSUM control chart are presented in this paper. The proposed explicit formulas were easy to calculate and code and required much less computational time to execute than NIE. Thus, we suggest that they can be applied to real-world applications for a variety of processes in finance, economics, industrial manufacturing, etc. Moreover, the proposed explicit formulas for the ARL can be extended to other control charts.

Acknowledgments

The research was funding by Thailand Science Research and Innovation Fund, and King Mongkut's University of Technology North Bangkok Contract no. KMUTNBBasicR-64-02.

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